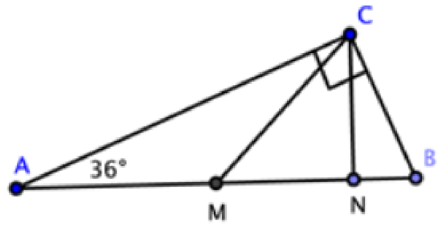
Solutions to short-answer questions

1



AB can be considered as a diameter of the circle centre M passing through C
$MC = MA$ (radii of the circle)
$\angle ACM = 36^\circ$ (isosceles triangle)
$\angle CMN = 72^\circ$ (sum of two interior angles is equal to the opposite exterior angle)
$\angle MCN = (180 - 72 - 90)^\circ = 18^\circ$

2 a Theorem 1: $y = \dfrac{140^\circ}{2} = 70^\circ$

$$\text{Theorem } 4: x + y = 180^\circ$$
$$x = 180^\circ - 70^\circ = 110^\circ$$

b Name the quadrilateral *ABCD*, in which y is at A and x is at B.
Let P be the point of intersection of AC and BD.

In triangle XCD,

$$\angle CDX = 180^\circ - 50^\circ - 75^\circ$$
$$= 55^\circ$$
$$\angle BCD = 90^\circ$$

(angle subtended by a diameter)

In triangle BCD,

$$x = \angle BDC$$
$$= 180^\circ - 90^\circ - 55^\circ$$
$$= 35^\circ$$
$$y = x = 35^\circ (\text{angles subtended by the same arc})$$

c Angles in the same segment are equal:
$x = 47^\circ$

$y = 53^\circ$

z is the exterior angle of either triangle.

Using the left triangle:

$$z = x + 53^\circ$$
$$= 47 + 53^\circ$$
$$= 100^\circ$$

d First note that $y = x$.

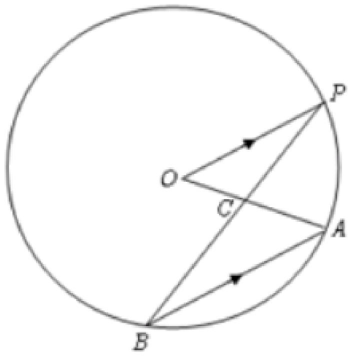
Consider the concave quadrilateral containing the 30° angle.

Its angles are 30°, $180^\circ - 70^\circ = 110,\ x + 70^\circ$ and $x + 70^\circ$, using supplementary angles, vertically opposite angles and exterior angles of a triangle.

$$x + 70 + x + 70 + 110 + 30 = 260$$
$$2x + 280 = 260$$
$$x = 40^\circ$$

$$y = 40^\circ$$
$$z = 180 - (x + 70)$$
$$= 70^\circ$$

3



a Using angles on arc AP,
$\angle POA = 2\angle CBA$

Using alternate angles, $\angle POA = \angle CAB$

$\therefore \quad \angle CAB = 2\angle CBA$

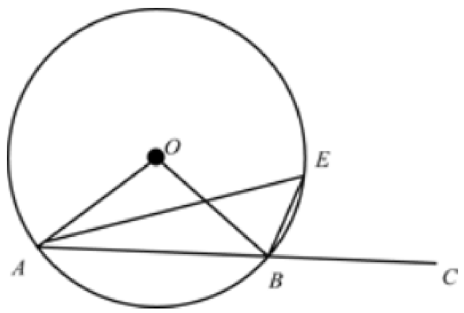
b Using angles on arc AP,
$\angle POA = 2\angle CBA$

Using alternate angles, $\angle OPC = \angle CBA$

Using the exterior angle of triangle OCP,

$$\begin{aligned}\angle PCA &= \angle POC + \angle OPC \\ &= \angle POA + \angle OPC \\ \angle PCA &= 2\angle CBA + \angle CBA \\ &= 3\angle CBA\end{aligned}$$

4



$\angle OBC = \angle OAB + \angle AOB$ (exterior angle of triangle AOB)

$\angle OBC = \angle OAB + \angle AOB$ (exterior angle of triangle AEB)

$$\angle BAE = \frac{1}{2}\angle OAB$$

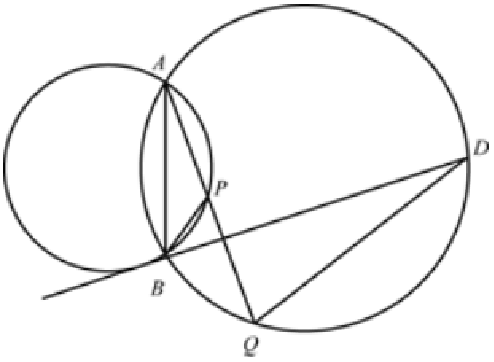
$$\angle BEA = \frac{1}{2}\angle AOB \text{ (angles on arc } AB)$$

$$\therefore \quad \angle EBC = \frac{1}{2}(\angle OAB + \angle AOB)$$

$$= \frac{1}{2}\angle OBC$$

i.e. EB bisects $\angle OBC$.

5



$\angle PBD = \angle BAP$ (alternate segment)

$\angle BAP = \angle BDQ$ (angles on BQ)

$\therefore \quad PBD = \angle BDQ$

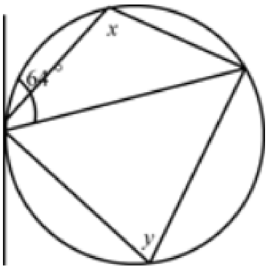
These are alternate angles on BP and QD.

$\therefore \quad BP$ is parallel to QD.

6 **a** The base angle of the isosceles triangle is 57° (alternate segment theorem)

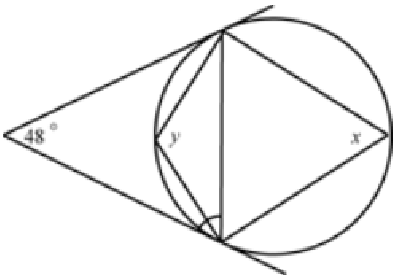
$$x = 180^\circ - 57^\circ - 57^\circ$$
$$= 66^\circ$$

b Make a construction as shown below.



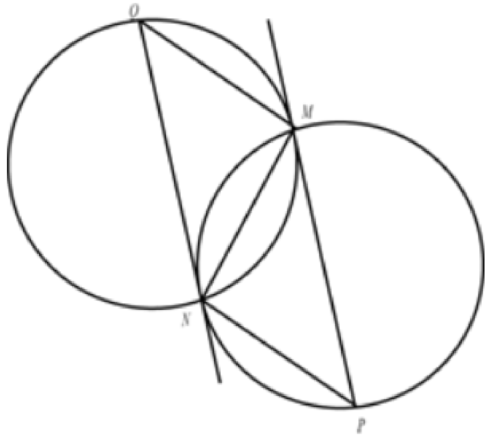
$$y = 64^\circ \text{ (alternate segment theorem)}$$
$$x = 180^\circ - 64^\circ$$
$$= 116^\circ \text{ (cyclic quadrilateral)}$$

c Make a construction as shown below.



$$\text{Marked angle} = \frac{180^\circ - 48^\circ}{2}$$
$$= 66^\circ$$
$$x = 66^\circ \text{ (alternate segment)}$$
$$y = 180^\circ - 66^\circ$$
$$= 114^\circ \text{ (cyclic quadrilateral)}$$

7



Consider triangles MNQ and NPM.

$\angle MQN = \angle NMP$ (alternate segment)

$\angle MNQ = \angle NPM$ (alternate segment)

$\therefore$ the triangles are similar and $\dfrac{MN}{Np} = \dfrac{QM}{MN}$.

Cross multiplying gives $MN^2 = NP \cdot QM$

8 $AE \cdot EB = CE \cdot ED$

$15 \times 5 = 25ED$

$ED = DE$

$= 3 \text{ cm}$

Solutions to multiple-choice questions

1 **B** In isosceles triangle ABD,

$\angle ABD = \angle ADB$

$= \dfrac{180^\circ - 70^\circ}{2} = 55^\circ$

$\angle ACD$ is subtended by the same arc, so $\angle ACD = 55^\circ$.

2 **A** In quadrilateral *OAPB*,

$\angle OAP = \angle OBP = 90^\circ$

$\angle APB = 360^\circ - 150^\circ - 90^\circ - 90^\circ$

$= 30^\circ$

The angle subtended at the circumference on minor arc AB is

$\dfrac{150^\circ}{2} = 75^\circ$.

This angle is opposite Q in a cyclic quadrilateral.

$\therefore \angle AQB = 180^\circ - 75^\circ = 105^\circ$

3 **E** There are multiple ways to solve this problem.

$\angle OAB = 68^\circ$

$\angle BAT = 90^\circ - 68^\circ = 22^\circ$

$\angle ABT = 180^\circ - 20^\circ - 68^\circ = 92^\circ$

$\angle ATB = 180^\circ - 22^\circ - 92^\circ = 66^\circ$

4 **A** $\angle BAC = 60^\circ$

$\text{Reflex } \angle BOC = 360^\circ - 120^\circ$

$= 240^\circ$

In quadrilateral *ABOC*,

$\angle ABO = 360^\circ - 240^\circ - 42^\circ - 60^\circ$

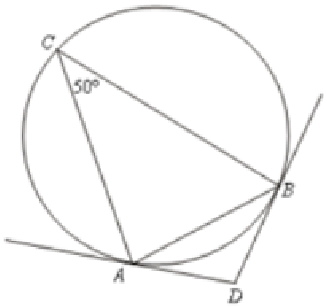
$= 18^\circ$

5 **C** $\angle DAB = 180^\circ - 65^\circ = 115^\circ$

Corresponding angles on parallel lines

$\therefore \angle CBE = 115^\circ$

6 **A**



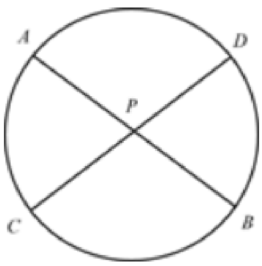
$\angle BAD = 50^\circ$ (alternate segment theorem)

$\angle ABD = 50^\circ$ (alternate segment theorem)

In triangle ABD,

$$\angle ADB = 180^\circ - 50^\circ - 50^\circ$$
$$= 80^\circ$$

7 **C**



$$AP \cdot PB = CP \cdot PD$$
$$12 \times 6 = 2PD$$
$$PD = 36 \text{ cm}$$

8 **B**

$$NB = 13 - 5 = 8 \text{ cm}$$
$$NQ = PN$$
$$AN \cdot NB = PN \cdot NQ$$
$$= PN^2$$
$$18 \times 8 = PN^2$$
$$PN = \sqrt{144} = 12 \text{ cm}$$
$$PB^2 = 12^2 + 8^2 = 208$$
$$PB = \sqrt{208}$$
$$= \sqrt{16 \times 13}$$
$$= 4\sqrt{13} \text{ cm}$$

9 **A** In triangle BAX,

$$\angle BAX = 180^\circ - 40^\circ - 105^\circ$$
$$= 35^\circ$$

Angles are subtended by the same arc

$\angle XSC = \angle BAX = 35^\circ$

10 **A** $\angle CDA = 90^\circ$ (angle subtended by a diameter)

In triangle ACD,

$$\angle CAD = 180^\circ - 90^\circ - 25^\circ$$
$$= 65^\circ$$

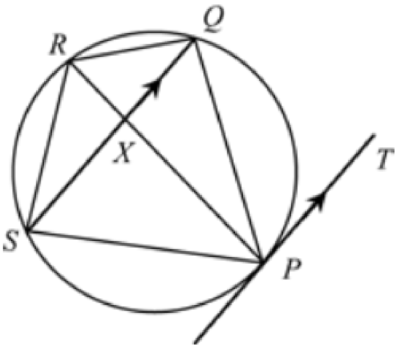
$\angle CBD = \angle ACD = 65^\circ$
$\angle BCD = 180^\circ - 75^\circ = 105^\circ$

In triangle BCD,

$\angle BDC = 180^\circ - 105^\circ - 65^\circ$
$= 10^\circ$

Solutions to extended-response questions

1 a Let PT be the tangent.

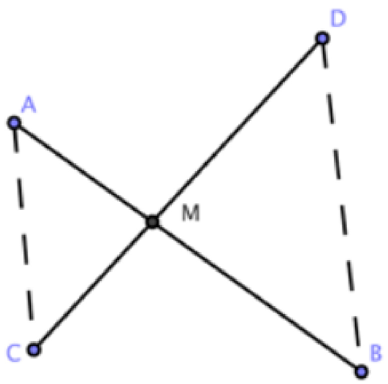


$\angle PSQ = \angle QPT$ (alternate segment)
$\angle PQS = \angle QPT$ (alternate angles)
Since $\angle PSQ = \angle PQS$, triangle PQS is isosceles

with $PQ = PS$, as required to prove.

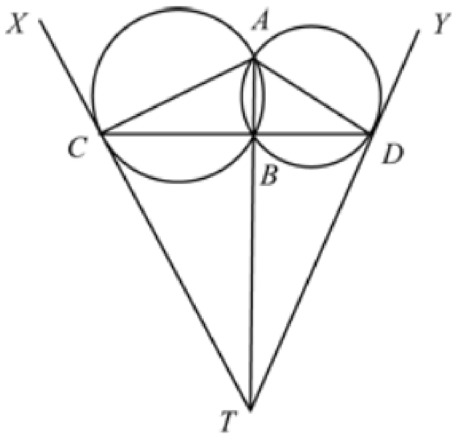
b $\angle PRS = \angle PQS$ (same segment)
$\angle PRQ = \angle QPT$ (alternate segment)
$= \angle PQS$ (alternate angles)
$= \angle PRS$
$\angle QRS = \angle PRQ + \angle PRS = 2\angle PRS$
Therefore PR bisects $\angle QRS$, as required to prove.

2



$AM \times BM = CM \times DM$ implies $\dfrac{AM}{CM} = \dfrac{DM}{BM}$
$\angle AMC = \angle BMD$
Hence $\triangle AMD \sim \triangle DMB$ (SAS). Hence $\angle CAM = \angle BDM$
Hence $ABCD$ is cyclic (converse of Theorem 2)

3 a



$$\angle XCA + \angle ACB + \angle BCT = 180^\circ \text{ (supplementary)}$$
$$\angle XCA = \angle CBA \text{ (alternate segment)}$$
$$\angle TCB = \angle CAB \text{ (alternate segment)}$$
$$\angle BCA + \angle CAB = \angle ABD \text{ (exterior angle of triangle)}$$
$$\angle YDA = \angle ABD \text{ (alternate segment)}$$
$$\angle YDA + \angle ADB + \angle BDT = 180^\circ$$
$$\therefore \quad \angle ABD + \angle ADB + \angle BDT = 180^\circ$$
$$\therefore \quad \angle BCA + \angle CAB + \angle ADB + \angle BDT = 180^\circ$$

But $\angle CAB = \angle TCB$

$$\therefore \quad \angle BCA + \angle TCB + \angle ADB + \angle BDT = 180^\circ$$
$$\therefore \quad \angle ACT + \angle ADT = 180^\circ$$

$\therefore$ $TCAD$ is a cyclic quadrilateral, as required to prove.

b

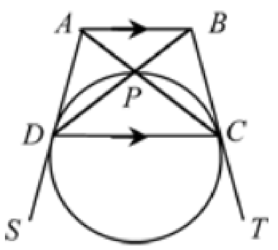
$$\begin{aligned} \angle TAC &= \angle BAC \\ &= \angle BCT \text{ (alternate segments in circle } ABC) \\ &= \angle DCT \\ &= \angle TAD \text{ (same segment in circle } ACTD) \end{aligned}$$
$$\therefore \quad \angle TAC = \angle TAD, \text{ as required to prove.}$$

c $TC^2 = TB \cdot TA$ and $TD^2 = TB.TA$ (tangent/secant theorem)

$$\therefore \quad TC^2 = TD^2$$
$$\therefore \quad TC = TD, \text{ as required to prove.}$$

4 a AD is a tangent to the circle CDP.

$$\begin{aligned} \angle BAC &= \angle ACD \text{ (alternate angles, } AB \parallel CD) \\ &= \angle DCP \\ &= \angle ADP \text{ (alternate segment)} \\ &= \angle ADB, \text{ as required to prove.} \end{aligned}$$



b

$$\begin{aligned} \angle BAP &= \angle BAC \\ &= \angle ADB \text{ (from a)} \\ &= \angle ADP \end{aligned}$$

$\therefore$ AB is a tangent to the circle ADP (alternate segment), with point of contact A, i.e. the circle ADP touches AB at the point A, as required to prove.

c Let $\angle BAC = x^\circ$, $\angle ABD = y^\circ$,

$$\begin{aligned} \therefore \quad \angle ACD &= x^\circ \text{ (alternate angles)} \\ \angle ADB &= x^\circ \text{ (alternate segment)} \\ \angle BDC &= y^\circ \text{ (alternate angles)} \\ \angle BCA &= y^\circ \text{ (alternate segment)} \\ \therefore \quad \angle ADC &= (x+y)^\circ \\ \text{and } \angle DCB &= (x+y)^\circ \\ \angle ABC + \angle DCB &= 180^\circ \text{ (co-interior angles)} \\ \therefore \quad \angle ABC + \angle ADC &= 180^\circ \end{aligned}$$

$\therefore$ $ABCD$ is a cyclic quadrilateral, as required to prove.

5 a i

$$\begin{aligned} \angle MSX &= \angle MSR \text{ (supplementary angles)} \\ &= 90^\circ \end{aligned}$$

Also $MP = MS$ (M midpoint of PS) and $\angle SMX = \angle PMQ$ (vertically opposite)

$\therefore$ triangles MPQ and MSX are congruent.

$$\begin{aligned} \text{Therefore } SX &= PQ \\ &= RS \text{ (opposite sides of a square)} \end{aligned}$$

$\therefore$ S is the midpoint of RX.

Also $\angle PSX = \angle PSR$

$$= 90^\circ \text{ (angle in a square)}$$

Therefore triangle XPS is congruent to triangle RPS.

$\therefore$ $XP = RP$ and triangle XPR is isosceles, as required to prove.

ii $PS = RS$ (sides of a square)

$\therefore$ triangle PRS is isosceles with $\angle RPS = 45^\circ$

$\therefore$ $\angle RPX = 90^\circ$

$\therefore$ $PX \perp OP$ and PX is a tangent to the circle at P, as required to prove.

b

$$\begin{aligned} \text{Area of trapezium} &= \text{area of square } PQRS + \text{area of triangle } PSX \\ &= 4^2 + \frac{1}{2} \times 4^2 \\ &= 16 + 8 \\ &= 24 \text{ cm}^2 \end{aligned}$$

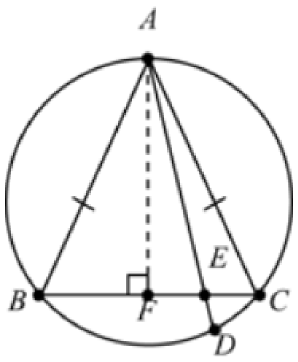
6 a Let AF be the perpendicular bisector of BC (since $AB = AC$).

$AB^2 = BF^2 + AF^2$ ① (Pythagoras' theorem)

$AE^2 = AF^2 + FE^2$ ② (Pythagoras' theorem)

① − ② yields

$$\begin{aligned} AB^2 - AE^2 &= BF^2 + AF^2 - AF^2 - FE^2 \\ &= BF^2 - FE^2 \\ &= (BE - FE)^2 - FE^2 \\ &= BE^2 - 2BE \cdot FE + FE^2 - FE^2 \\ &= BE^2 - 2BE \cdot FE \\ &= BE(BE - 2FE) \\ &= BE(BF + FE - 2FE) \\ &= BE(BF - FE) \\ &= BE(CF - FE) \text{ (since } BF = CF) \\ &= BE \cdot CE, \text{ as required to prove.} \end{aligned}$$



b Extend PT to meet the circle again at Q.

$\therefore \quad PT \cdot QT = AT \cdot BT$ (intersecting chords)

but $QT = PT$ since AB is a diameter

$\therefore \quad PT^2 = AT \cdot BT$

and, by the tangent/secant theorem

$$CP^2 = CA \cdot CB$$
$$\text{Also } CP^2 = CT^2 + PT^2 \text{ (Pythagoras'theorem)}$$
$$\therefore \quad CA \cdot CB = CT^2 + AT \cdot BT$$

$\therefore \quad CA \cdot CB - TA \cdot TB = CT^2$, as required to prove.